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SELF-CIRCUMFERENCE IN THE MINKOWSKI PLANE

Mostafa Ghandehari
Richard Pfiefer

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Self-Circumference in the Minkowski Plane

Mostafa Ghandehari
Naval Postgraduate School
Monterey, CA 93943

Richard Pfiefer
San Jose State University
San Jose, CA 95192

Abstract

Let $\sigma(n)$ denote the self-circumference of a regular polygon with n sides. It will be shown that $\sigma(n)$ is monotonically increasing from 6 to 2π if n is twice an odd number, and monotonically decreasing from 8 to 2π if n is twice an even number. Calculation of $\sigma(n)$ for the case where n is odd as well as inequalities for self-circumference of some irregular polygons are given. Properties of the mixed area of a plane convex body and its polar dual are used to discuss the self-circumference of some convex curves.

Introduction

Minkowski distance defined by means of a convex body was developed by Minkowski [12], and has applications in study of crystals [20] and differential equations [15]. Minkowski spaces are simply finite dimensional normed linear spaces. The articles by Busemann [2] and Petty [13] contain basic concepts for the study of Minkowskian geometry, as do Chapter 6 of Benson's book [1] and Chapter 4 of Valentine's book [22]; these last two books also contain useful background material from the theory of convex sets.

In this article, we deal with the perimeter of unit circles in the Minkowski plane, using the terminology of geometry of numbers, to give calculations and inequalities for the self-circumference of unit circles in the Minkowski plane. This work started with an attempt at specific calculations for the regular polygons, from which some inequalities grew.

By a plane convex body K we shall mean a compact, convex subset of the Euclidean plane having a non-empty interior. We shall take a "unit circle" K for the Minkowski plane to be a centrally symmetric convex body with its center at the origin in the Euclidean plane. The Minkowski distance $d(x,y)$ from x to y is defined by

$$(1) \quad d(x,y) = d_e(x,y)/r ,$$

where $d_e(x,y)$ is the Euclidean length from x to y , and r is the Euclidean radius of K in the direction of the vector $y-x$. See Figure 1. We will refer to the points of the Euclidean plane with this new metric as “the Minkowski plane”. The Minkowski length of a polygon path is obtained by adding the Minkowskian lengths of the corresponding line segments. The Minkowskian length of a curve is defined by taking supremum over all inscribed polygonal paths. The self-circumference of the unit circle K is the Minkowskian length of K measured with respect to K . In other words, the length of boundary of K using the metric induced by K is called the “self-circumference” of K and is denoted by $\sigma(K)$. Gołab [7] was apparently the first to prove that

$$(2) \quad 6 \leq \sigma(K) \leq 8 .$$

Equality is attained on the left if, and only if, K is affine image of a regular hexagon, and on the right if, and only if, K is a parallelogram. A proof of the above inequality is given by Schäffer [16]. His book [17], Geometry of Spheres in Normed Spaces, also contains a proof as well as historical background related to self-circumference. Chakerian and Talley [3]

establish a number of properties of self-circumference and raise interesting questions.

Some Definitions

Let K be a plane convex body with the origin as an interior point. For each angle θ , $0 \leq \theta < 2\pi$, we let $r(K, \theta)$ be the radius of K in direction $(\cos \theta, \sin \theta)$, so that the boundary of K has equation $r = r(K, \theta)$ in polar coordinates. The distance from the origin to the supporting line of K with outward normal $(\cos \theta, \sin \theta)$ is denoted by $h(K, \theta)$. This is the supporting function of K restricted to the Euclidean unit circle. Since K is convex, it has a well-defined unique tangent line at all but at most a countable number of points. We let $ds(K, \theta)$ represent the element of Euclidean arclength of the boundary of K at a point where the unit normal is given by $(\cos \theta, \sin \theta)$. Then the perimeter of K is given by

$$(3) \qquad L(K) = \int_0^{2\pi} h(K, \theta) d\theta \quad ,$$

while the Euclidean area of K is given by

$$(4) \qquad A(K) = \frac{1}{2} \int_0^{2\pi} h(K, \theta) \, ds(K, \theta) \quad .$$

The polar dual of K , denoted by K^* , is another plane convex body having the origin as an interior point and is defined in such a way that

$$(5) \quad h(K^*, \theta) = \frac{1}{r(K, \theta)} \quad \text{and} \quad r(K^*, \theta) = \frac{1}{h(K, \theta)} .$$

The mixed area $A(K_1, K_2)$ of two convex sets is defined by

$$(6) \quad A(K_1, K_2) = \frac{1}{2} \int_0^{2\pi} h(K_1, \theta) ds(K_2, \theta) .$$

It turns out that the mixed area is symmetric in its arguments. Eggleston [5] contains further properties of mixed areas. The following result due to Firey [6] will be used: The mixed area of a plane convex body and its polar dual is at least π .

The unit circle K of a Minkowskian plane is referred to as the indicatrix. Define the isoperimetrix to be that convex body T such that

$$(7) \quad h(T, \theta) = \frac{1}{r(K, \theta + \pi/2)} = h(K^*, \theta + \pi/2) .$$

(See Petty [13].) A centrally symmetric set is called a Radon curve if it coincides with the corresponding isoperimetrix. Pages 233, 234 of Benson [1] contain further properties of Radon curves. We now discuss the definition of self-circumference and give some properties.

If K is a centrally symmetric plane convex body, centered at the origin, then, because of (1) and the succeeding discussion, the self-circumference $\sigma(K)$ is given by

$$(8) \quad \sigma(K) = \int ds(K, \theta) / r(K, \theta + \pi/2) \quad .$$

If K is not necessarily centrally symmetric and z is any point interior to K , then positive and negative self-circumference of K relative to z are defined by

$$(9) \quad \sigma_+(K, z) = \int ds(K, \theta) / r(K, \theta + \pi/2)$$

and

$$(10) \quad \sigma_-(K, z) = \int ds(K, \theta) / r(K, \theta - \pi/2) \quad .$$

where the origin of the coordinate system is at z . Both $\sigma_+(K, z)$ and $\sigma_-(K, z)$ reduce to $\sigma(K)$ in case K is centrally symmetric with z as its center. Gołab [7] conjectured that $\sigma_{\pm}(K, z) \geq 6$, for all $z \in K$, and $\min_{z \in K} \sigma_{\pm}(K, z) \leq 9$. The latter conjecture was settled by Grünbaum [8]. The proof given in [9] of the lower bound appears to be in error. The paper of Sorokin [19] also is apparently in error, so that this question is still open.

If K_1 and K_2 are plane convex bodies with the origin as an interior point, then the length of the positively oriented boundary of K_1 with respect to K_2 is given by

$$(11) \quad \sigma_+(K_1, K_2) = \int ds(K_1, \theta) / r(K_2, \theta + \pi/2)$$

and the length of the negatively oriented boundary is given by

$$(12) \quad \sigma_-(K_1, K_2) = \int ds(K_1, \theta) / r(K_2, \theta - \pi/2) .$$

Schäffer [18] and independently later, Thompson [21], proved that for a centrally symmetric set $\sigma_+(K) = \sigma_-(K^*)$ and $\sigma_-(K) = \sigma_+(K^*)$. More generally Chakerian [4] used the concept of mixed areas to prove that

$$(13) \quad \sigma_+(K_1, K_2) = \sigma_-(K_2^*, K_1^*) \quad \text{and} \quad \sigma_-(K_1, K_2) = \sigma_+(K_2^*, K_1^*) .$$

Holmes and Thompson [11] give a definition of area and content in Minkowski spaces which implies that the surface of the unit ball and that of the dual ball are the same.

Polygons

The following theorem gives inequalities for the self-circumference of regular polygons with an even number of sides.

We shall later discuss the calculation of self-circumference for regular polygons with an odd number of sides and for some irregular polygons.

Theorem 1. Let $\sigma(n)$ denote the self-circumference of an affine image of a regular polygon with n sides. Then $\sigma(n)$ is monotonically increasing from 6 to 2π if n is twice an odd number, and monotonically decreasing from 8 to 2π if n is twice an even number. Furthermore, 6 and 8 are the only two rational values assumed by these families of polygons.

Proof. The affine invariance of self-circumference implies that we only need to consider regular polygons. Assume n is twice an odd number. Inscribe the polygon in a circle. There are an odd number of sides on each side of a diameter. Thus one of the sides is parallel to a diameter. By simple trigonometry it can be shown that the Minkowskian length of a side is equal to $2 \sin \pi/n$. Thus

$$(14) \quad \sigma(n) = 2n \sin \pi/n .$$

Monotonicity of $\frac{\sin x}{x}$ and the fact that $\frac{\sin x}{x} < 1$ for $0 < x < \pi/2$ implies that $\sigma(n) < 2\pi$. Since $\sin \pi/n$ assumes rational values only for $n = 2$ and $n = 6$ (see Pólya and Szegő [14], page 144, problems 197.1 and 197.5), it follows that 6 is

the only rational value attained. By using a diameter perpendicular to two parallel sides it can be shown that

$$(15) \quad \sigma(n) = 2n \tan \pi/n$$

for the case when n is twice an even number. Monotonicity of $\frac{\tan x}{x}$ and the fact that $\tan \pi/n$ is only rational for $n = 4$ implies the second case. ■

The self-circumference of a regular n -gon, where n is odd, is given by

$$(16) \quad \sigma(n) = (2n \tan \pi/n) (\cos \pi/n)$$

It is interesting to look for polygons with self-circumference equal to 7. The self-circumference of the polygon in Figure 2 is given by

$$(17) \quad \sigma(K) = 8(1 + \tan^2 \theta/1 + \tan \theta) .$$

This is a convex function of θ symmetric with respect to $\theta = \pi/8$, attaining its minimum at $\theta = \pi/8$, corresponding to a regular octagon. We can choose a value of θ such that $\sigma(K) = 7$. Similar formulas can be derived for polygons obtained by truncating the corners of regular polygons.

Chakerian and Talley [3] discuss the self-circumference $\eta_{\pm}(K)$ at the centroid and give some estimates and conjectures. For example they make the following conjecture.

Conjecture. The minimum value of $\eta_{+}(Q)$, as Q ranges over all convex quadrilaterals, is equal to the minimum value of

$$(18) \quad a + \frac{6}{(a-1)(4-a)} + \frac{6}{(a^2-4a+6)} \quad , \quad 2 < a < 3 \quad .$$

which is approximately 7.8201 and is attained for $a = a_0 \approx 2.6317$.

They use numerical techniques and their data indicate that the minimum value $\eta_{+}(P)$, as P ranges over all convex pentagons, is no more than 6.835. They also raise the following question.

Question. What is the least upper bound of $\eta_{+}(K)$, as K ranges over all plane convex bodies?

They give an example of a trapezoid to show that $\sigma_{+}(K,z)$ and $\sigma_{-}(K,z)$ do not assume their minimum at the same point, thus answering a question posed by Hammer [10]. The following two theorems give inequalities for self-circumference of a quadrangle and a trapezoid.

Theorem 2. The self-circumference $\sigma_{+}(Q,z)$ of a quadrilateral with respect to the point of intersection of the diagonals is

at least 8, with equality if and only if the quadrilateral is a parallelogram.

Proof. Consider a quadrilateral Q with vertices A, B, C, D (see Figure 3). Let z be the point of intersection of the diagonals. Let a, b, c, d be the Euclidean lengths from z to A, B, C and D respectively. Then by using similar triangles we obtain

$$(19) \quad \sigma_+(Q, z) = \frac{a+c}{c} + \frac{b+d}{d} + \frac{a+c}{a} + \frac{b+d}{b} .$$

Thus

$$(20) \quad \sigma_+(Q, z) = 4 + \frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} \geq 8 ,$$

where the last inequality follows from the arithmetic-geometric mean inequality. Equality is attained if and only if $a = c$, $b = d$, which implies Q is a parallelogram. ■

Theorem 3. The self-circumference $\sigma_+(Q, 0)$ of a trapezoid Q with respect to the midpoint of one of the diagonals is at least 8, with equality if and only if Q is a parallelogram.

Proof. Consider a trapezoid Q with vertices A, B, C and D . See Figure 4. In the figure \overline{OH} is parallel to \overline{AD} , \overline{OE} is

parallel to \overline{AB} , \overline{OF} is parallel to \overline{BC} , and \overline{OG} is parallel to \overline{CD} .
By using similar triangles we obtain

$$\begin{aligned}
 (21) \quad \sigma_+(Q,0) &= \frac{\overline{AB}}{\overline{OE}} + \frac{\overline{BC}}{\overline{OF}} + \frac{\overline{CD}}{\overline{OG}} + \frac{\overline{DA}}{\overline{OH}} \\
 &= \frac{\overline{AB}}{\overline{OE}} + \frac{\overline{CD}}{\overline{OG}} + 4 = 2\left(\frac{\overline{AB}}{\overline{CD}} + \frac{\overline{CD}}{\overline{AB}}\right) + 4 ,
 \end{aligned}$$

where we have used the fact that $\overline{CD} = 2\overline{OE}$, $\overline{AB} = 2\overline{OG}$, $\overline{BC} = 2\overline{OF}$ and $\overline{AD} = 2\overline{OH}$. The arithmetic-geometric mean inequality and equation (21) imply $\sigma_+(Q,0) \geq 8$ with equality if and only if $\overline{CD} = \overline{AB}$, which gives a parallelogram. ■

Calculation of self-circumference of some simple polygons is interesting. For example, for the pentagon P in Figure 5, $\sigma_+(P,z) = 7$. The self-circumference $\sigma_+(K,z)$ for the polygons in Figures 6, 7, 8, 9 and 10 assumes the values 6, 7, 8, 9 and 10 respectively. By moving z along the altitude of the triangle in Figure 10, other values of $\sigma_+(K,z)$ will be assumed, approaching infinity as z gets close to the vertex. The self-circumference of the family of hexagons obtained by truncation of an equilateral triangle at the corners is given by

$$(22) \quad \sigma(K) = 9 - \frac{3t}{2} , \quad 0 \leq t \leq 2 .$$

The value $\sigma(K,z) = 9$ is obtained when $t = 0$ and the value $\sigma(K,z) = 6$ when $t = 2$. The self-circumference of the family of

polygons in Figure 12 varies continuously from 6 to 8. In particular, for equation (22) and Figure 12, hexagons can be found with self-circumferences equal to 2π . Thus we obtain Minkowski unit circles with self-circumferences equal to that of the Euclidean unit circle.

Curves

In the following we use properties of mixed areas of a plane convex body and its polar dual to discuss self-circumference of some convex curves. The following theorem shows that the self-circumference of a plane convex body with four-fold symmetry is at least 2π .

Theorem 4. Let K be a centrally symmetric plane convex body centered at the origin. Assume $r(K, \theta)$ is an equation of the boundary of K in polar coordinates. Assume $r(K, \theta) = r(K, \theta + \frac{\pi}{2})$, $0 \leq \theta \leq 2\pi$. That is, K has four-fold symmetry. Then the self-circumference satisfies $\sigma(K) \geq 2\pi$.

Proof. Using the definition given in (8), and four-fold symmetry, we obtain

$$\begin{aligned}
 (23) \quad \sigma(K) &= \int ds(K, \theta) / r(K, \theta + \pi/2) \\
 &= \int ds(K, \theta) / r(K, \theta) .
 \end{aligned}$$

By the property of polar dual given in (5) and the property of mixed areas given in (6), it follows that

$$\begin{aligned}
 (24) \quad \sigma(K) &= \int ds(K, \theta) / r(K, \theta) = \int h(K^*, \theta) ds(K, \theta) \\
 &= 2A(K^*, K).
 \end{aligned}$$

Firey's result [6] states that the mixed area of a plane convex body and its polar dual is at least π . By using this and equation (24), it follows that $\sigma(K) \geq 2\pi$. \square

Regular polygons with the number of sides equal to twice an even number have four-fold symmetry. Thus the result in Theorem 1 is a special case of the above theorem. The unit circles of 1_p spaces satisfy four-fold symmetry and thus their self-circumference is at least 2π , with the value of 2π for the case $p = 2$.

Recall that the isoperimetrix T was defined by (7). The following theorem gives the Minkowskian length of a plane convex body with respect to the isoperimetrix.

Theorem 5. Let K be a plane convex body. Assume T is the isoperimetrix, that is, the polar dual rotated 90 degrees. Then $\sigma_+(K, T) = 2A(K)$, where $A(K)$ is the Euclidean area.

Proof. By the definition given in (11) we obtain

$$\sigma_+(K, T) = \int ds(K, \theta) / r(T, \theta + \pi/2) .$$

By the definition of isoperimetrix, $r(T, \theta + \pi/2) = r(K^*, \theta)$.

Thus,

$$(25) \quad \sigma_+(K, T) = \int ds(K, \theta) / r(K^*, \theta) = \int h(K, \theta) ds(K, \theta) = 2A(K) ,$$

where we have used (4) and (5) giving the Euclidean area and the property of polar dual. \square

If K is a Radon curve, then it coincides with its isoperimetrix. Thus the self-circumference of a Radon curve is equal to twice its Euclidean area. We conclude by proving the following theorem, concerning the length of a Euclidean unit circle with respect to a convex curve K .

Theorem 6. Let K be a plane convex body. Assume B is the Euclidean unit circle. Then the length of B with respect to K is equal to the Euclidean length of the polar dual of K . That is, $\sigma_+(B, K) = L(K^*)$.

Proof. By the result of Chakerian given in (13) we obtain

$$(26) \quad \sigma_+(B, K) = \sigma_-(K^*, B^*) = \sigma_-(K^*, B) .$$

Assuming that the polar dual of K is calculated at the center of the Euclidean unit circle B , it follows that $\sigma_+(B,K) = L(K^*)$. \square

In the particular case where K is a square with vertices at $(\pm 1, 0)$, $(0, \pm 1)$, the Minkowski distance is the same as the so-called Taxicab metric. The polar dual is a square with sides parallel to the axis. See Figure 13. Thus the length of a Euclidean unit circle in Taxicab metric is the same as the Euclidean length of the circumscribed square which is 8, and thus we have finally squared the circle.

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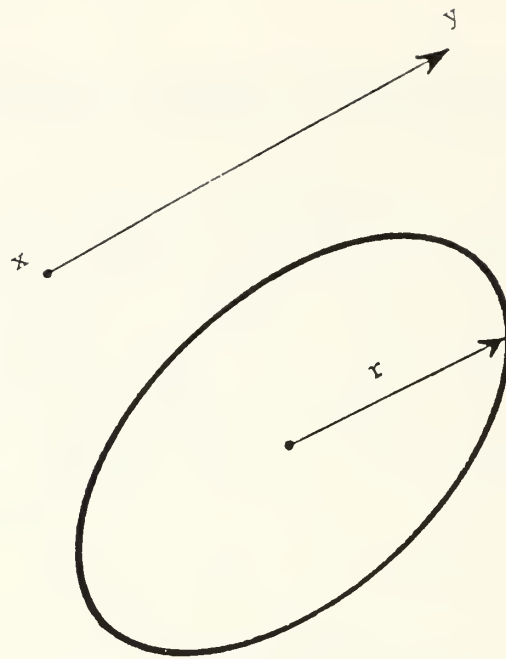


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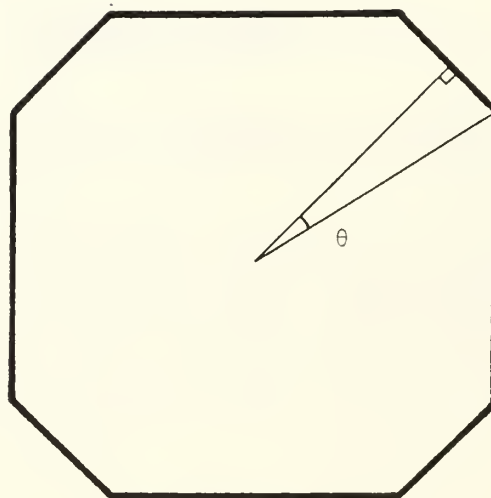


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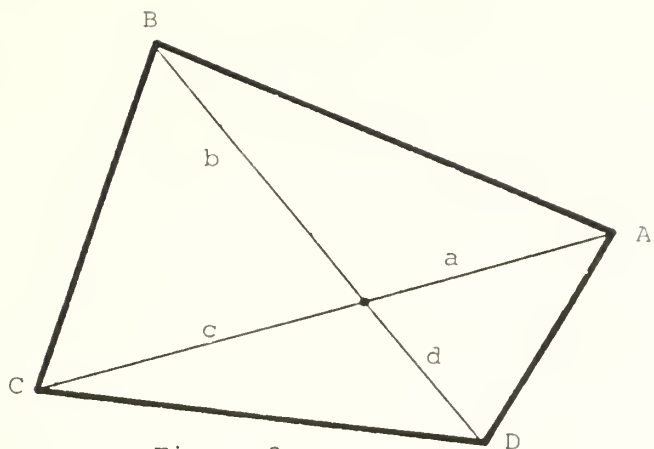


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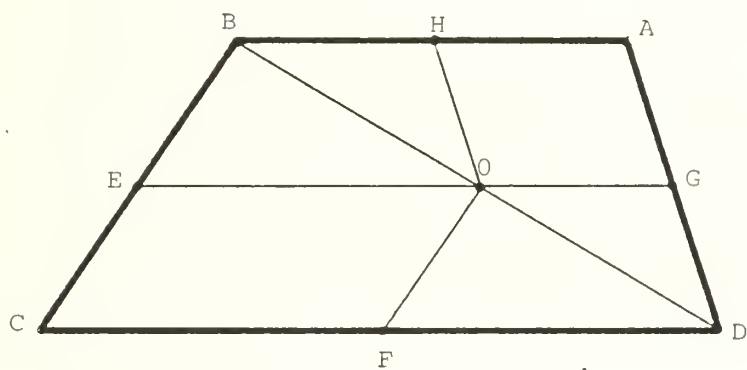


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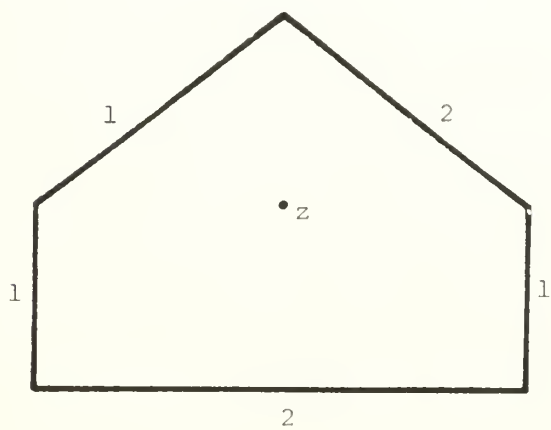


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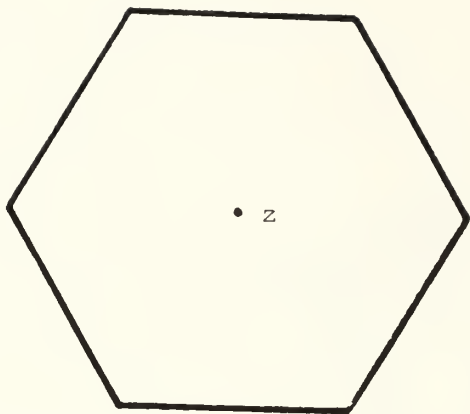


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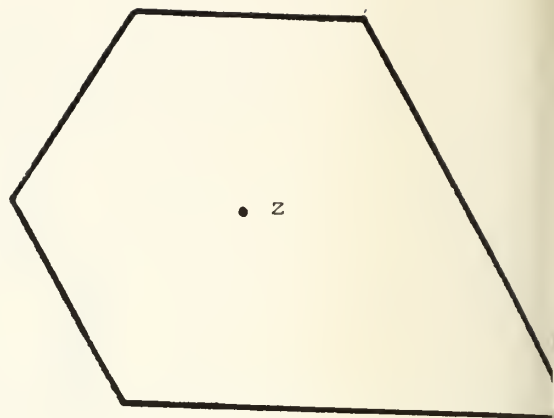


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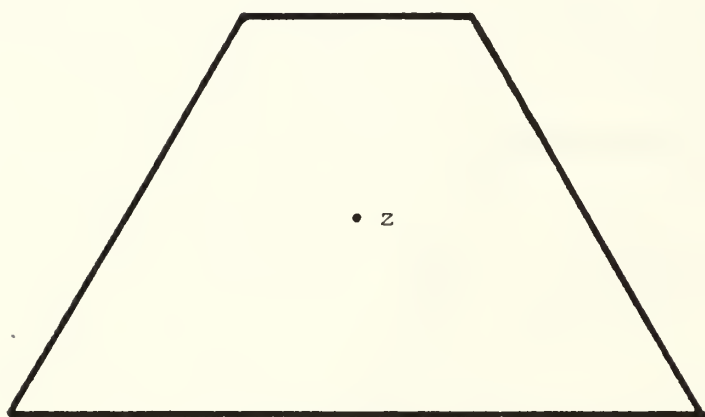


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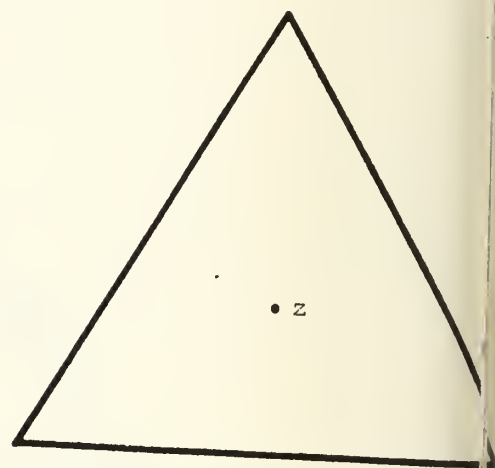


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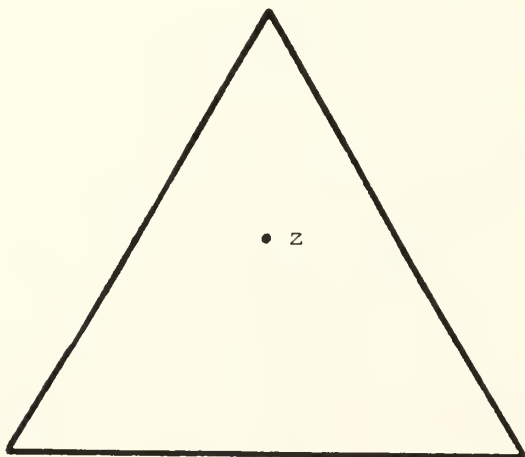


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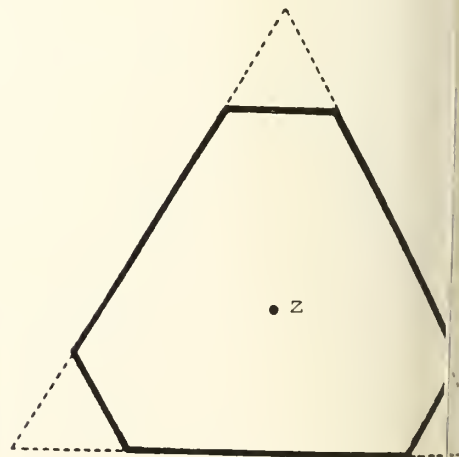


Figure 11

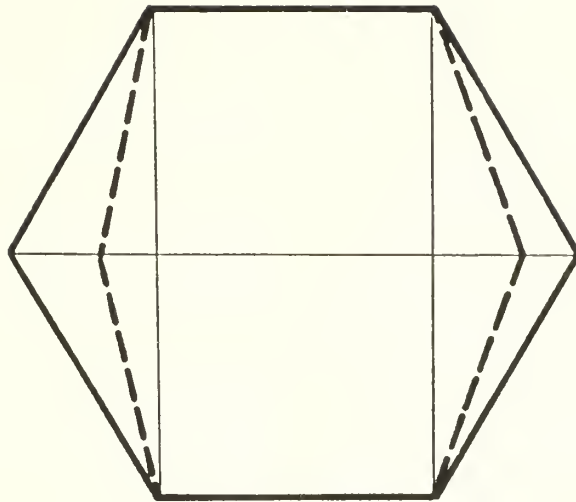


Figure 12

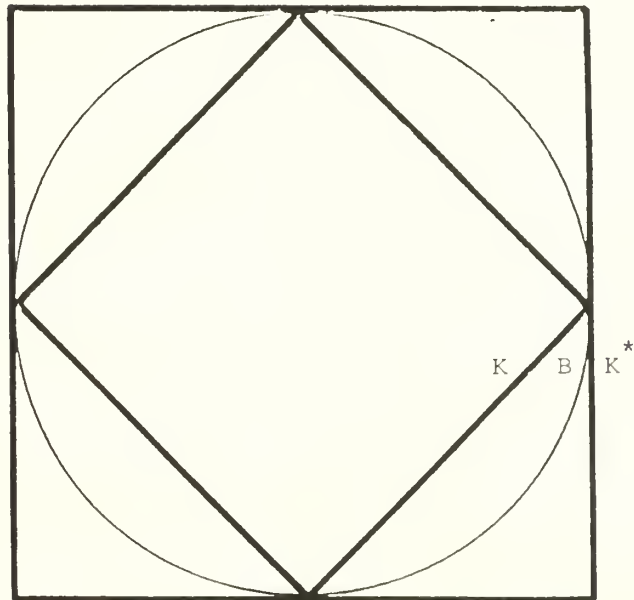


Figure 13

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Department of Mathematics
University of California
Davis, CA 95616

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